Scaling law for the maximum Lyapunov characteristic exponent of infinite product of random matrices

1986 J. Phys. A: Math. Gen. 19 L425

(http://iopscience.iop.org/0305-4470/19/8/003)

J. Phys. A: Math. Gen. 19 (1986) L425-L428. Printed in Great Britain

LETTER TO THE EDITOR

Scaling law for the maximum Lyapunov characteristic exponent of infinite product of random matrices

Giorgio Parisi† and Angelo Vulpiani‡

† Dipartimento di Fisica, Università di Roma II, 'Tor Vergata' 00173 Roma, Italy and INFN Sezione di Roma, Italy

‡ Dipartimento di Fisica, Università di Roma I, 'La Sapienza' P le A Moro 2, 00185 Roma, Italy and GNSM-CNR Unità di Roma, Italy

Received 10 February 1986

Abstract. We present a simple derivation for the scaling behaviour of the maximum Lyapunov characteristic exponent λ of infinite product of symplectic random matrices. The considered random matrices depend on a parameter ε and $\lambda = 0$ for $\varepsilon = 0$. We obtain $\lambda \propto \varepsilon^{\beta}$ with either $\beta = \frac{1}{2}$ or $\beta = \frac{2}{3}$ depending on the probability distribution of the matrix elements. The results are in agreement with a previous numerical simulation.

Recently the problem of the scaling behaviour of the maximum Lyapunov exponent λ for the infinite product of random symplectic matrices has been studied numerically. Benettin (1984) and Paladin and Vulpiani (1986) studied the behaviour of λ for infinite product of symplectic matrices with the following form:

$$A_{\varepsilon}(k) = \left(\begin{array}{c|c} \mathbb{1} & \mathbb{1} \\ \hline \varepsilon\hat{a}(k) & \mathbb{1} + \varepsilon\hat{a}(k) \end{array}\right) \tag{1}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix, $\hat{a}$ is a symmetric random matrix whose non-zero elements are only $a_{1,N}$, $a_{N,1}$ and such that $|i-j| \leq 1$.

For λ defined by

$$\lambda = \lim_{M\to\infty} \frac{1}{M} \ln\left[\left|\left(\prod_{k=1}^{M} A_{\varepsilon}(k)\right)\zeta(0)\right| \Big/ |\zeta(0)|\right] \tag{2}$$

it is trivial to see that at $\varepsilon = 0$, $\lambda = 0$; the authors mentioned above found

$$\lambda \propto \varepsilon^{\beta} \tag{3}$$

with $\beta = \frac{1}{2}$ if the mean value of $a_{ij} \neq 0$ and $\beta = \frac{2}{3}$ if the mean value of $a_{ij} = 0$. The interest in the problem is due to the fact that if one considers a symplectic map

$$\begin{aligned} \boldsymbol{q}(n+1) &= \boldsymbol{q}(n) + \boldsymbol{p}(n) \\ \boldsymbol{p}(n+1) &= \boldsymbol{p}(n) + \varepsilon \nabla F[\boldsymbol{q}(n+1)] \end{aligned} \tag{4}$$

where $\nabla = (\partial/\partial q_1, \ldots, \partial/\partial q_N)$, the linearised evolution of (4) which gives λ is given by a matrix of the form (1). So replacing the 'true' linearised evolution of map (4) with a product of random matrices of the form (1) one has a crude, but not trivial, approximation of the dynamics. Moreover the map (4) can be considered as the Poincaré map of an $(N+1)$-dimensional Hamiltonian system.

0305-4470/86/080425+04$02.50

In addition, symplectic random matrices (but with a form different from (1)) are involved in solid state problems with disorder (Pichard and Sarma 1981, Derrida and Gardner 1984). Also in this case in the limit of weak disorder one has in the case of 2×2 matrices a scaling law with $\beta=\frac{2}{3}$ (Derrida and Gardner 1984).

Now we derive the scaling law (3) with $\beta=\frac{1}{2}$ and $\beta=\frac{2}{3}$. Introducing

$$A_0=\left(\begin{array}{c|c}\mathbb{1} & \mathbb{1}\\ \hline 0 & \mathbb{1}\end{array}\right) \qquad b_\varepsilon(k)=\left(\begin{array}{c|c}0 & 0\\ \hline \varepsilon\hat{a}(k) & \varepsilon\hat{a}(k)\end{array}\right)$$

we have

$$B_M=\prod_{k=1}^{M}A_\varepsilon(k)=A_0^M+\sum_{i+j=M-1}A_0^ib_\varepsilon(i)A_0^j+\sum_{i+j+k=M-2}A_0^ib_\varepsilon(i)A_0^jb_\varepsilon(i+J)A_0^k$$
$$+\ldots+\sum_{i_1+i_2+\ldots+i_n=M-(n-1)}[A_0^{i_1}b_\varepsilon(i_1)A_0^{i_2}b_\varepsilon(i_1+i_2)\ldots A_0^{i_n}]+\ldots. \tag{5}$$

It is easy, noting that

$$A_0^n=\left(\begin{array}{c|c}\mathbb{1} & n\mathbb{1}\\ \hline 0 & \mathbb{1}\end{array}\right)$$

to compute the leading contribution to the average of an element of the matrix B_M given by the term $\Sigma_{i_1+\ldots+i_{n+1}=M-n}[A_0^{i_1}\ldots A_0^{i_{n+1}}]$ in (5), if $\overline{a_{ij}}=\alpha>0$ (where $\overline{(\cdot)}$ indicates the average over the non-zero elements of the matrix) and if $N\geqslant 3$, this gives rise to

$$3^n\alpha^n\varepsilon^n\sum_{i_1+i_2+\ldots+i_{n+1}=M-n}i_1,i_2,\ldots,i_{n+1}\simeq\frac{(3\alpha\varepsilon)^n}{(2n+1)!}M^{2n+1}. \tag{6}$$

In (6) the term 3^n is due to the fact that only 3 elements on a row of the matrix $\hat{a}$ are non-zero and the sum $\Sigma_{i_1+\ldots+i_n=M-(n-1)}i_1,i_2,\ldots,i_n$ is approximated by $\int_{x_1+\ldots+x_n=M}x_1,x_2,\ldots,x_n\,dx_1,\ldots,dx_n$.

From (6) we obtain for large M

$$\langle|B_M\zeta(0)|\rangle\sim\exp L(1)M \tag{7}$$

with

$$L(1)=\sqrt{3}\alpha^{1/2}\varepsilon^{1/2} \tag{7'}$$

where $\langle(\cdot)\rangle$ is the average of the probability distribution of the random matrices.

Note that $L(1)$ is not exactly equal to λ (see Benzi *et al* 1985); we recall that the generalised Lyapunov exponent $L(q)$ defined (for large M) as

$$\langle|B_M\zeta(0)|^q\rangle\sim\exp L(q)M \tag{8}$$

in the general case presents some deviation from the linear law

$$L(q)=\lambda q. \tag{9}$$

The relation (9) holds exactly only in the limit of no intermittency so the estimation $\lambda=L(1)$ (or $\lambda=\frac{1}{2}L(2)$ in the case $\overline{a_{ij}}=0$) is an approximation. Rigorously it is an upper bound because $L(q)/q$ must increase with q (see Benzi *et al* 1985). However it is reasonable to assume that the intermittency changes only the constant in front of ε^β and not the power laws. This is verified in the numerical computation.

The case with $\overline{a_{ij}}=0$ and $\overline{a_{ij}^2}=\gamma$ need a slightly different computation. Let us consider

$$B_M^+B_M=\left[(A_0^+)^M+\sum_{i+j=M+1}(A_0^+)^i b_\varepsilon^+(i)(A_0^+)^j + \sum_{i+j+k=M-2}(A_0^+)^i b_\varepsilon^+(i)(A_0^+)^j b_\varepsilon^+(i+j)(A_0^+)^k+\ldots\right] \times\left[A_0^M+\sum_{i+j=M-1}A_0^i b_\varepsilon(i)A_0^j+\sum_{i+j+k=M-2}A_0^i b_\varepsilon(i)A_0^j b_\varepsilon(i+j)A_0^k+\ldots\right]. \tag{10}$$

Note that because $\overline{a_{ij}}=0$ and moreover $\hat{a}(n)$ is independent of $\hat{a}(m)$, then if $m\neq n$ in the mean value of an element of $B_M^+B_M$ only the terms

$$(A_0^+)^M A_0^M+\sum_{i+j=M-1}(A_0^+)^i b_\varepsilon^+(i)(A_0^+)^j A_0^j b_\varepsilon(i)A_0^i + \sum_{i+j+k=M-2}(A_0^+)^i b_\varepsilon^+(i)(A_0^+)^j b_\varepsilon^+(i+j)(A_0^+)^k A_0^k b_\varepsilon(i+j)A_0^j b_\varepsilon(i)A_0^i + \ldots \tag{11}$$

give a non-zero contribution.

Because of the independence of $\hat{a}(k)$ and the fact that $\overline{a_{ij}}=0$ one has that the nth term in (11) is

$$3^n\gamma^n\varepsilon^{2n}\sum_{i_1+\ldots+i_{n+1}=M-N} i_1^2\ldots i_{n+1}^2\simeq 2\frac{(6\gamma\varepsilon^2)^n}{(3n+2)!}M^{3n+2}. \tag{12}$$

For equation (12) the same remarks hold as for equation (6). From (12) we obtain

$$\langle(\boldsymbol{\zeta}(0),B_M^+B_M\boldsymbol{\zeta}(0))\rangle=\langle|B_M\boldsymbol{\zeta}(0)|^2\rangle\sim\exp L(2)M \tag{13}$$

with

$$L(2)=6^{1/3}\gamma^{1/3}\varepsilon^{2/3}. \tag{13'}$$

Therefore, in the approximation $\lambda=L(q)/q$ we obtain

$$\lambda=\begin{cases}\sqrt{3}\alpha^{1/2}\varepsilon^{1/2} & \text{if } \overline{a_{ij}}=\alpha>0\\ \frac{1}{2}6^{1/3}\gamma^{1/3}\varepsilon^{2/3} & \text{if } \overline{a_{ij}}=0.\end{cases} \tag{14}$$

The numerical simulations give the same power laws with the constants in front of ε^β smaller than those in (14) (let us recall that because of the concavity of $L(q)$, (14) is an upper bound). In the case $\overline{a_{ij}}>0$ the difference in the constant between (14) and numerical computation is O(5%) and in the other case O(15%). The scaling laws (7′) and (13′) are in agreement with numerical computation with a precision of 1%.

Part of this work was carried out during our visit to the L D Landau Institute for Theoretical Physics in Moscow; we thank the USSR Academy of Science for this opportunity. We are grateful to O N Dorokhov for useful discussions. A particular acknowledgment is due to G Paladin for fruitful suggestions and his help in the numerical computations.

References

Benettin G 1984 *Physica* **13D** 211
Benzi R, Paladin G, Parisi G and Vulpiani A 1985 *J. Phys. A: Math. Gen.* **18** 2157
Derrida B and Gardner E 1984 *J. Physique* **45** 1283
Paladin G and Vulpiani A 1986 *J. Phys. A: Math. Gen.* **19** in press
Pichard J L and Sarma G 1981 *J. Phys. C: Solid State Phys.* **14** L127